CALCULATION OF CHARGE DENSITY DISTRIBUTION PARAMETERS OF SILICON ISOTOPE $^{30}_{14}$Si FOR 3PG-MODEL

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Abstract

In this research, three parameters Fermi model of $^{30}_{14}Si$ are change to three parameters Gaussian model of $^{30}_{14}Si$ in which the parameters of charge density distribution are calculated. The differential scattering cross section of two-body nuclear system is studied in this work. The charged form factors are fitted for 3pG charge distribution model based on the 3pF model which is used as references for this study. The charge form factor which connects scattering cross section and the size of nucleus was discussed. From our calculations, it is found that our fitted charge density distribution and form factor are in good agreement with that of 3pF model. In particular, the purpose of the research shows that the charge form factor is an observable manifestation of an internal structure and shape of the nuclei.

Keywords*:* Differential scattering cross section, charge form factor, charge density distribution.

Introduction

Nuclear charge density distribution parameters are obtained from elastic electron scattering. They can be expressed in various form factors depending upon nuclear models. Nuclear models are two parameters Fermi model, three parameter Fermi model, three parameter gaussian model, harmonic oscillator model and uniform gaussian model. In this research, the charge form factor for silicon isotope nucleus $^{30}_{14}$Si will be studied for 3PG model with the help of reference 3pF charge distribution model [Schneuwly.R., H.Vuilleumier & J.L Walter, et.al, (1974)]. Silicon is used for electronic devices as a semiconductor. At still higher energies, quark degrees of freedom and nucleon resonances may play an important role [Williams.W.S.C, (1991)]. In this research, the charge form factor for silicon isotope nucleus $^{30}_{14}$Si will be studied for 3PG model with the help of reference 3pF charge distribution model. The form factor of a nucleus is of fundamental importance for our understanding of its internal structure. Form factor give information about charge distributions and size of nucleus [Angeli. L. & K.P Marinova, (2013)].

Rutherford Differential Scattering Cross Section

In this section, we can obtain the differential cross section from an asymptotic form of the solution of the Schrödinger equation. Let us first focus on the determination $f(\theta,\phi)$; it can also be obtained from the solution of the equation, which is in the form of

$$-\frac{\hbar^2}{2\mu}\vec{\nabla}^2 \quad \psi(\vec{r}) \quad + \quad \hat{V}(\vec{r})\psi(\vec{r}) \quad = \quad E\psi(\vec{r})$$

$$\vec{\nabla}^2\,\psi(\vec{r}) + \frac{2\mu}{\hbar^2}\left[E - \hat{V}(\vec{r})\right]\psi(\vec{r}) \;= 0 \tag{1}$$

This equation (1) in turn can be rewritten as follow;

$$(\vec{\nabla}^2 + k^2)\psi(\vec{r}) = \frac{2\mu}{\hbar^2}V(\vec{r})\psi(\vec{r}) \tag{2}$$

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The general solution to this above equation consists of a sum of two components; homogeneous (incident plane wave) and particular solution to equation (2). We can express it in terms of The Green's function. Thus, the general solution of equation (2) is given by

$$\Psi(\vec{r}) = \psi_{inc}(\vec{r}) + \frac{2\mu}{\hbar^2}\int G(\vec{r}-\vec{r}')V(\vec{r}')\,\psi(\vec{r}')d^3r' \tag{3}$$

Where $G(\vec{r}-\vec{r}')$ is a green's function corresponding to the operator on left hand side on equation (2). The function $G(\vec{r}-\vec{r}')$ is obtained by solving the point source equation;

$$(\vec{\nabla}^2 + k^2)G(\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}') \tag{4}$$

Where $\delta(\vec{r}-\vec{r}')$ is Dirac delta function. $G(\vec{r}-\vec{r}')$ and $\delta(\vec{r}-\vec{r}')$ are given by the Fourier transform as follow;

$$G(\vec{r}-\vec{r}') = \frac{1}{(2\pi)^3}\int e^{i\vec{q}.(\vec{r}-\vec{r}')}G(\vec{q})d^3\vec{q} \tag{5}$$

$$\delta(\vec{r}-\vec{r}') = \frac{1}{(2\pi)^3}\int e^{i\vec{q}.(\vec{r}-\vec{r}')}d^3\vec{q} \tag{6}$$

We can write using equation, the scattering amplitude and differential scattering cross section in the first-Born approximation as follow;

$$f(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\int \overrightarrow{\int_{inc}}^{\prime 3\prime} e^{-i\vec{k}.\vec{r}'}$$

$$f(\theta,\phi) = \frac{2\mu}{\hbar^2}\frac{Ze^2}{\alpha^2 + [2k\sin(\frac{\theta}{2})]^2}$$

For the pure Coulomb's potential, the range $\alpha = 0$ and then, we can write as follow;

$$f(\theta,\phi) = \frac{2\mu}{\hbar^2}\frac{Ze^2}{4k^2\sin^2(\frac{\theta}{2})}$$

Since $\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$, we made by squaring of both sides to the above equation. Then we get, the differential scattering cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 = \frac{4\mu^2Z^2e^4}{\hbar^4(2^4k^4\sin^4(\frac{\theta}{2}))}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \frac{4\mu^2Z^2e^4}{\hbar^4q^4} \tag{7}$$

The above equation (7) is the Rutherford differential scattering cross section.

Form Factor F(q)

In an electron elastic scattering process with a target nucleus, we have presented about the non-relativistic differential cross section (Rutherford). For a target nucleus having an extended finite size, the differential cross section differs from that of the Rutherford by a factor called form factor F(q) such as

$$\left(\frac{d\sigma}{d\Omega}\right)_{extended\,charge\,distribution} = \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford}|F(q)|^2 \tag{8}$$

Thus, the form factor indicates the effect of nuclear size upon the differential cross section, where q is the momentum transfer of the scattering process and $\vec{q} = \vec{k}_0 - \vec{k}$. When $|\vec{k}_0| = |\vec{k}| = k$, which can be express as$q = 2k\, sin(\frac{\theta}{2})$. The form factor is known as the Fourier transform of the charge density distribution and it can be expressed as follow;

$$F(q) = \frac{1}{Z}\int \rho_{ch}(r) e^{i\vec{q}.\vec{r}} d^3\vec{r} \tag{9}$$

The verification of the above equation (9) will be given later in the next section. The form factor plays an important role because it is the most important link between experimental observation and theoretical analysis. In an experiment, the form factor is the direct result of a cross section measurement. From the theoretical side, charge density distribution $\rho_{ch}(r)$ is a solution of the Schrodinger equation, which can be considered as the following relation.

$$\frac{d\sigma}{d\Omega_{Exp}} \rightarrow |F(q)| \Leftrightarrow |F(q)| \leftarrow \frac{d\sigma}{d\Omega_{Theory}} \leftarrow \rho_{ch}(r) \leftarrow \Psi(r) \leftarrow \text{Schrödinger Equation}$$

Relation between Charge Form Factor and Density Distribution

Now we will give the derivation of equation (9). Let us compute the scattering of an electron by a spherically symmetric nucleus having a finite size. The screened Coulomb potential between an electron and a point charge nucleus is

$$V(x) = -\frac{Ze^2}{x} e^{-x/a}. \tag{10}$$

However, we now consider the interaction between the electron and nucleus having a charge distribution. Thus, the screened Coulomb potential V(x) at the position of the electron "x" consists of contributions from the entire nucleus. An infinitesimal volume element $d\vec{r}$ contains a charge $dq = Ze\rho_{ch}(r)d\vec{r}$ which gives a contribution of

$$V(x) = -\frac{Ze^2}{y} e^{-y/a}\rho_{ch}(r)d\vec{r}, \quad \text{where } \vec{y} = \vec{x} - \vec{r}$$

$$V(x) = -Ze^2 \int \rho_{ch}(r) \frac{e^{-y/a}}{y} d\vec{r} \tag{11}$$

the first-Born approximation for a scattering amplitude f (q) is given by

$$f(q) = -\frac{m}{2\pi\hbar^2}\int V(x)\, e^{i\vec{q}.\vec{x}} d\vec{x} \tag{12}$$

Where, $q = 2k\, sin(\theta/2)$ is the momentum transfer of the elastic scattering process with scattering angle. By substituting the value of V(x) of equation (11), into the equation (12), we obtain the following form,

$$f(q) = \frac{mZe^2}{2\pi\hbar^2}\int\int \rho_{ch}(r) \frac{e^{-y/a}}{y} d\vec{r}\, e^{i\vec{q}.\vec{x}} d\vec{x} \tag{13}$$

By using $\vec{x} = \vec{r} + \vec{y}$ yields,

$$f(q) = \frac{mZe^2}{2\pi\hbar^2}\int\int \rho_{ch}(r) \frac{e^{-y/a}}{y} d\vec{r}\, e^{i\vec{q}.(\vec{r}+\vec{y})} d\vec{x}$$

$$f(q) = \frac{mZe^2}{2\pi\hbar^2}\int e^{i\vec{q}.\vec{r}}\rho_{ch}(r)d\vec{r}\int \frac{e^{-y/a}}{y} e^{i\vec{q}.\vec{y}} d\vec{x} \tag{14}$$

For fixed r, $d\vec{x}$ can be replaced by $d\vec{y}$.Then

$$f(q) = \frac{mZe^2}{2\pi\hbar^2}\int e^{i\vec{q}.\vec{r}}\rho_{ch}(r)\, d\vec{r}\int \frac{e^{-y/a}}{y} e^{i\vec{q}.\vec{y}} d\vec{y} \tag{15}$$

In the above equation (3.47), the integral term $\int \frac{e^{-y/a}}{y} e^{i\vec{q}.\vec{y}} d\vec{y}$ is evaluated as;

$$\int \frac{e^{-y/a}}{y} e^{i\vec{q}.\vec{y}} d\vec{y} = \frac{4\pi}{q^2+(1/a^2)} \tag{16}$$

If a >>1, equation (3.48) becomes the below form,

$$\int \frac{e^{-y/a}}{y} e^{i\vec{q}.\vec{y}} d\vec{y} = \frac{4\pi}{q^2}$$

Therefore, the equation (15) for the scattering amplitude f (q) can be written as follow;

$$f(q) = \frac{mZe^2}{2\pi\hbar^2} \int e^{i\vec{q}.\vec{r}} \rho_{ch}(r)\, d\vec{r} \times \frac{4\pi}{q^2}$$

and then, we get the scattering amplitude f (q) is in the form of;

$$f(q) = \frac{4\pi mZe^2}{2\pi\hbar^2 q^2} \int e^{i\vec{q}.\vec{r}} \rho_{ch}(r)\, d\vec{r} \tag{17}$$

Since, $\frac{d\sigma}{d\Omega} = |f(q)|^2$, we made by squaring of both sides to equation (17), Then we get,

$$\frac{d\sigma}{d\Omega} = |f(q)|^2 = \frac{4m^2Z^2e^4}{\hbar^4 q^4} \left| \int e^{i\vec{q}.\vec{r}} \rho_{ch}(r) d\vec{r} \right|^2 \tag{18}$$

However, the differential scattering cross section for electron scattering by a point nucleus (Rutherford scattering) is known to be previous section, which is in the form of;

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \frac{4m^2Z^2e^4}{\hbar^4 q^4} \tag{19}$$

By comparing equations (19) which is the differential scattering cross section for the target nucleus is considered as a point-charge and (18) the target nucleus is considered as with extended charge distribution. In these cases, the factor term is appeared and we can express as; $\left(\frac{d\sigma}{d\Omega}\right)_{experiment} = \left(\frac{d\sigma}{d\Omega}\right)_{\substack{(extended\\ charge distribution)}} = \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford(point charge)} |F(q)|^2$

$$|F(q)|^2 = \frac{\left(\frac{d\sigma}{d\Omega}\right)_{Experiment}}{\left(\frac{d\sigma}{d\Omega}\right)_{Ruther(point-charge)}} \tag{20}$$

The form factor is in the form of

$$F(q) = \int e^{i\vec{q}.\vec{r}} \rho_{ch}(r)\, d\vec{r} \tag{21}$$

If the normalization for form factor is chosen to be $F(q = 0) = 1$, the form factor can be written as;

$$F(q) = \frac{1}{Z} \int e^{i\vec{q}.\vec{r}} \rho_{ch}(r) d\vec{r} \tag{22}$$

The form factor $F(q)$ is the Fourier transform of the charge density distribution and the pervious equation (9) has been verified. The form factor $F(q)$ can also be expressed as the following form,

$$F(q) = \frac{1}{Z} \int e^{i\vec{q}.\vec{r}} \rho_{ch}(r) d^3\vec{r}$$

$$= \frac{1}{Z} \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} e^{iqr\cos\theta} \rho_{ch}(r) r^2 dr \sin\theta\, d\theta d\varphi \tag{23}$$

$$= \frac{2\pi}{Z} \int_{r=0}^{\infty} r^2 \rho_{ch}(r) \frac{e^{iqr} - e^{-iqr}}{iqr} dr = \frac{4\pi}{Z} \int_{r=0}^{\infty} r^2 \rho_{ch}(r) \frac{\sin(qr)}{qr} dr \tag{24}$$

$$F(q) = \frac{4\pi}{Z} \int_{r=0}^{\infty} r^2 \rho_{ch}(r)\, j_0(qr) dr \tag{25}$$

Nuclear Charge Density Distributions

Nuclear charged density distribution is obtained by taking the Fourier inverse of the form factor. The form factor is expressed as the form of equation (21). By multiplying the equation (21) with $\int e^{-i\vec{q}.\vec{r}'}\,d\vec{q}$ in both sides, and then

$$\int F(q)\, e^{-i\vec{q}.\vec{r}'}\,d\vec{q} = \int\int e^{i\vec{q}.\vec{r}} e^{-i\vec{q}.\vec{r}'}\,\rho_{ch}(r)\,d\vec{r}d\vec{q}$$

$$= \int^{\int d\vec{r}d\vec{q}} e^{i\vec{q}.\left(\vec{r}-\vec{r'}()\right)_{ch}(r)} = \int^{\int d\vec{r}} (2\pi)^3\delta\left(\vec{r}-\vec{r'}()_{ch}(r)\right)$$

$$\int F(q)\, e^{-i\vec{q}.\vec{r}'}\,d\vec{q} = (2\pi)^3\rho_{ch}(r')$$

$$\rho_{ch}(r) = \frac{1}{(2\pi)^3}\int F(q) e^{-i\vec{q}.\vec{r}}d\vec{q} \tag{26}$$

From an electron scattering experiment, the form factor can be deduced by the following equation,

$$|F(q)|^2 = \frac{\left(\frac{d\sigma}{d\Omega}\right)_{experiment}}{\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford}} \tag{27}$$

If the form factor $F(q)$ is known for all q, the inverse Fourier transform might be computed in order to get the charge density distribution, $\rho_{ch}(r)$ from the following equation

$$\rho_{ch}(r) = \frac{1}{(2\pi)^3}\int_{q=0}^{\infty} F(q) e^{-i\vec{q}.\vec{r}}d\vec{q}. \tag{28}$$

However, this is impossible because the form factor can be measured only in a limited range of momentum transfer "q". Thus, the simplest analysis is to construct a model of the charge distribution. Such models have parameters which can be adjusted until the calculated form factor provides the best fit to the available experimental measurements.

$$F(q)|_{calculated} = \frac{4\pi}{Z}\int_{r=0}^{\infty} r^2\rho_{ch}^{model}(r)\frac{\sin(qr)}{qr}dr. \tag{29}$$

Results and Discussion

Charge Density Distribution Parameters and Form Factor for 3pG Model

The charge density distribution parameters for 3pF (three parameters Fermi) model can be obtained from the reference data of Atomic and Nuclear Tables 36. [H.De.Vries, C. W. De Jager, & C. De Vries, (1987)]. We used the charge density distributions for the 3pF model as $\rho_{ch}(r) = \rho_0(1 + wr^2/c^2)/(1 + \exp((r - c)/z))$ and for form factor $F(q)|_{calculated} = \frac{4\pi}{Z}\int_{r=0}^{\infty} r^2\rho_{ch}^{model}(r)\frac{\sin(qr)}{qr}dr$. Then we constructed the form factor and charge density distribution profiles by the help of GFORTRAN CODE Program for that 3pF model using the reference data Table.1, these profiles are shown in figure 1 and 2. In this numerical calculation, number of integration steps N is taken to be 2000, step interval dr = 0.005. To find the charge density distribution parameters for the 3pg model, we first looked for the difference value of the charge density distribution parameters between the 3pf and 3pg models of another nucleus which were already known in references paper. Based on the value of the difference obtained, we considered the appropriate values to fit the charge density distribution parameters for the desired 3pg model. It was estimated that the program was run over 2,000 times with various possible values. From our multiple observations, we record the closest possible values was listed in table. Among these obtained values, we choices one of the best parameters set to use the set of the charge density distribution parameters for the 3pg model.

After that we have fitted the charge density distribution parameters for 3PG model that is three parameters Gaussian model by using the formula, $\rho_{ch}(r) = \rho_0(1 + wr^2/c^2)/(1 + \exp((r^2 - c^2)/z^2))$ and we fitted these parameters based on 3pF profile as reference. We have shown best set of fitting parameters in Table 2. We plotted the Figure (4.3) to (4.4) by the used of our best set fitted data. And we compared these 3pG profiles which is used our fitted data set with already known 3pF profiles for both density distribution and form factor. From the comparison of these figures, we get the corresponding fitted parameters for 3pG model.

Finally, we can clearly say that our finding parameters are good in agreement to use as a charged density distribution parameter for 3pG model; because from the figure 3 to 4, the reference and fitted curve are the same as a single line in the momentum transfer range of 3fm for both charge density distribution and charge form factor.

Table.1 Reference charge density distribution parameters for 3pF model of silicon isotope

Nucleus	C	Z	W
	Radius Parameter	**Skin Thickness Parameter**	**Adjustable Parameter**
$^{30}_{14}Si$	3.252	0.553	-0.078

Table.2 Finding out the charge density distribution parameters for 3pG model of silicon isotope

Nucleus	C	Z	W
	Radius Parameter	**Skin Thickness Parameter**	**Adjustable Parameter**
$^{30}_{14}Si$	2.731	2.034	0.199

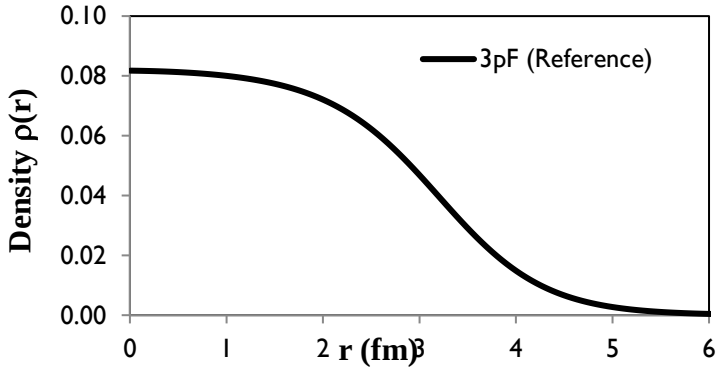

Figure 1: Charge density distribution for 3pF model of $^{30}_{14}$Si versus the radius

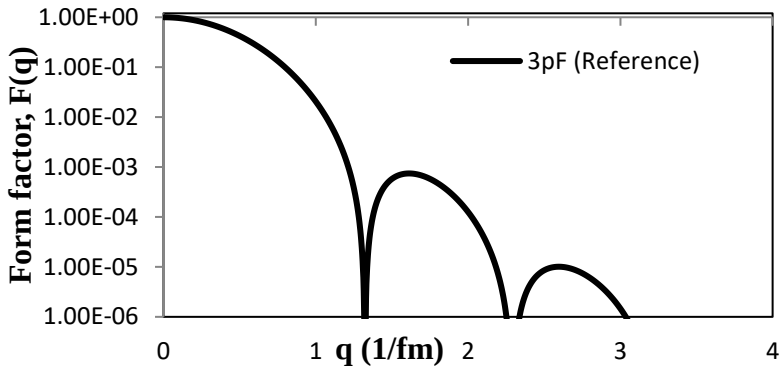

Figure 2: Charge form factor for 3pF model of $^{30}_{14}Si$ versus the momentum transfer

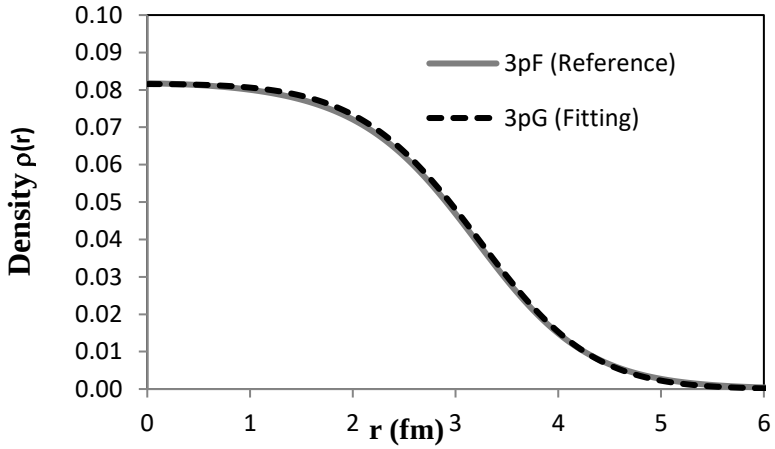

Figure 3: Comparison of the charge density distribution for reference 3pF model (solid curve) and 3pG model (dash curve) of silicon isotope $^{30}_{14}Si$

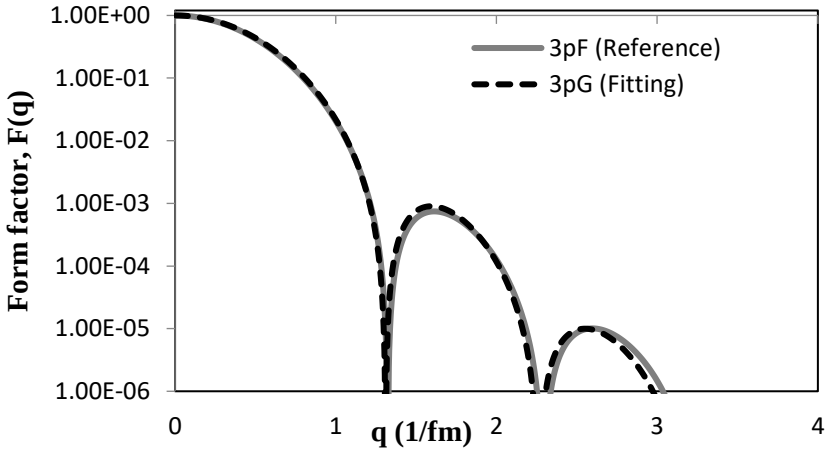

Figure 4: Comparison of charge form factor for reference 3pF model (solid curve) and 3pG model (dash curve) of silicon isotope $^{30}_{14}Si$

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